Three-dimensional viscous wakes

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The velocity fields of three-dimensional viscous wakes are examined with the use of the boundary-layer approximations, Oseen's linearization of the convective terms, and the assumption of constant fluid properties. Transform methods yield solutions for general types of initial conditions. As an illustration, the axial velocity distribution of a wake whose initial isovels (lines of constant velocity) are of elliptic shape and their decay to axial symmetry are demonstrated. Both laminar and turbulent flows are considered.

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1. Introduction

The aim here is to make an initial assessment of some three-dimensional effects in viscous free mixing of wake-like or jet-like flows. Our considerations are limited to flows in which the Prandtl boundary-layer approximations may be made. As a result the cross-flow velocities v, w are assumed small with respect to the axial velocity u, and the cross-wise pressure variations are assumed to have a negligible influence on the axial momentum balance. In fact, streamwise pressure gradients are completely neglected in the axial momentum equation in a first approximation. Clearly this approach may be formalized by means of appropriately constructed series expansions in terms of a small parameter which is a function of the characteristic Reynolds number. Higher approximations may be obtained thereby. However, in this paper the formalism and higher approximations are not considered. Rather the equations, simplified to the first order, are obtained by familiar order of magnitude reasoning of the boundary-layer type. A related discussion is given by Bloom (1961).

To achieve additional simplicity, flows are considered wherein decrements or increments with respect to a uniform inviscid velocity u_e are sufficiently small to permit the linearization of the convective terms in the equations of motion. This linearization is in the familiar spirit of Oseen and others. Jets in stationary ambients are excluded here although analogous procedures may be applied to them as well. Finally, steady flow and constancy of fluid properties are postulated.

From a mathematical standpoint, the governing equations exhibit 'quasi-elliptic' behaviour in the cross-planes and 'parabolic' behaviour axially. That is, one real characteristic exists, whereas two are imaginary. The linearized equation governing the axial velocity is seen to be uncoupled from those governing the cross-flow components. On the other hand, the two cross-flow momentum equations, in which the cross-wise pressure gradients are retained are formed into two equations of Poisson type. Transform methods can be used to produce solutions of the governing equations for general types of initial conditions. Both laminar and turbulent flows are considered.

2. Analysis

The following equations are assumed to apply

continuity: $$v_y + w_z = \bar{u}_x; \tag{1}$$

momentum (linearized boundary-layer equations):

$$\rho u_e \bar{u}_x = \mu \nabla_1^2 \bar{u}; \quad p = p(y,z), \tag{2}$$

$$\rho u_e v_x = \mu \nabla_1^2 \bar{v} - p_y, \tag{3}$$

$$\rho u_e w_x = \mu \nabla_1^2 \bar{w} - p_z, \tag{4}$$

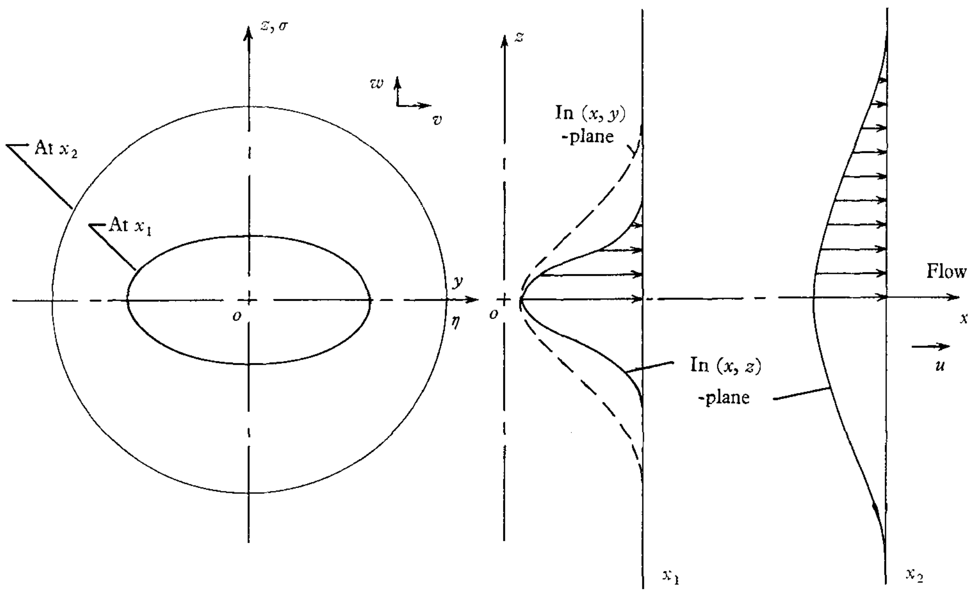

FIGURE 1. Schematic diagram of axial flow field. Typical isovels are shown on the left and velocity defect profiles $(u_e - u)$ on the right.

where ρ denotes density, p pressure, μ the coefficient of viscosity, $\bar{u}$ the axial velocity defect $(u_e - u)$, $\nabla_1^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$, subscripts x, y, z denote partial differentiation with respect to the indicated variable, subscript e denotes the conditions at the edge of the viscous layer, and the co-ordinates and velocity components are defined in figure 1. Equations (1) to (4) govern either laminar or turbulent flows. However, for the latter μ is interpreted to be the turbulent eddy viscosity which is assumed to be at most a function of x.

The solution for the velocity components is first developed in general terms and then applied to a particular problem, a wake whose axial velocity isovels (lines of constant velocity) are of elliptic shape. The distribution of axial velocity is considered first.

Axial velocity

Equation (2) is recast as follows

$$\phi_s = \phi_{\eta\eta} + \phi_{\sigma\sigma} \tag{5}$$

with
$$\frac{x-x_c}{L} = \int_0^s \left[\frac{\mu}{\mu_t}\right]^j ds, \quad \eta = \frac{y}{L}\left(\frac{Lu_e}{\nu}\right)^{\frac{1}{2}}, \quad \sigma = \frac{z}{L}\left(\frac{Lu_e}{\nu}\right)^{\frac{1}{2}}, \tag{5a}$$

and
$$\phi = (u_e - u)/(u_e - u_{0_c}), \tag{5b}$$

where $j = 0$ for laminar flows, $j = 1$ for turbulent flows, L defines a constant characteristic length, μ_t is the eddy coefficient of viscosity, and subscripts c and 0 denote conditions at an initial station and conditions along the s axis, respectively. Equation (5) indicates that in terms of the transformed variables the analysis for laminar and turbulent flows are identical.

The appropriate boundary conditions are

$$\text{at} \quad s = 0, \quad \phi = \phi_c(\eta, \sigma), \tag{6a}$$

$$\text{as} \quad s \to \infty, \quad \phi \to 0, \tag{6b}$$

and
$$\text{as} \quad \eta \to \infty \quad \text{or} \quad \sigma \to \infty, \quad \phi = 0. \tag{6c}$$

Use of double Fourier Transforms yields the following solution of (5)

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z(a,b)\exp\{-(a^2+b^2)s - i(a\eta + b\sigma)\}\,da\,db, \tag{7}$$

where
$$Z(a,b) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_c e^{i(a\eta+b\sigma)}\,d\eta\,d\sigma. \tag{8}$$

In the special case where it can be assumed that ϕ_c is separable, that is

$$\phi_c = \phi_{1_c}(\eta)\,\phi_{2_c}(\sigma), \tag{9}$$

equation (7) can be rewritten as follows

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_1(a)\,e^{-a^2s - ia\eta}\,d\eta \int_{-\infty}^{\infty} Z_2(b)\,e^{-b^2s - ib\sigma}\,d\sigma, \tag{10}$$

where $Z_1(a)$ and $Z_2(b)$ are the usual one-dimensional transforms of the functions $\phi_{1_c}(\eta)$, and $\phi_{2_c}(\sigma)$, respectively.

Cross-flow velocities

The governing equations for the cross-flow velocities are

$$\text{continuity:} \quad v_y + w_z = (u_e - u_{0_c})\,\phi_x; \tag{11}$$

$$\text{momentum:} \quad u_e v_x = -\frac{1}{\rho}p_y + \nu\nabla_1^2 v, \tag{12}$$

$$u_e w_x = -\frac{1}{\rho}p_z + \nu\nabla_1^2 w, \tag{13}$$

where ϕ_x is assumed to be known from prior analysis. Differentiation of (12) and (13) with respect to z and y, respectively, yields the vorticity equation

$$\Omega_s = \Omega_{\eta\eta} + \Omega_{\sigma\sigma}, \tag{14}$$

where $\Omega = v_z - w_y$. The vorticity equation is identical in form to (5) and is subject to the same form of boundary conditions (6). Thus, the solution is

$$\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(m,n)\exp\{-(m^2+n^2)s - i(m\eta+n\sigma)\}\,dm\,dn, \tag{15}$$

where
$$T(m,n) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Omega_c(\eta,\sigma)\,e^{i(m\eta+n\sigma)}\,d\eta\,d\sigma \tag{16}$$

and $\Omega = \Omega_c(\eta,\sigma)$ at $s = 0$. Since Ω and ϕ are now determined, the components v and w are obtained from the following relations,

$$\nabla_1^2 v = (u_e - u_{0_c})\,\phi_{xy} + \Omega_z, \tag{17a}$$

$$\nabla_1^2 w = (u_e - u_{0_c})\,\phi_{xz} - \Omega_y, \tag{17b}$$

and are amenable to solution by classical techniques.

Example: elliptic wake

(*a*) Laminar flow ($j = 0$). At an initial station ($s = 0$), let

$$\phi_c = \frac{u_e - u_c}{u_e - u_{0_c}} = \exp\left\{-\left[\eta^2 + \left(\frac{\sigma}{\epsilon}\right)^2\right]\right\} \tag{18}$$

where ϵ is a pure constant, which lies in the range $0 < \epsilon \leqslant 1$, and defines the eccentricity of the isovels. Subscripts e, c and 0 denote free stream conditions, conditions at an initial station, and conditions at the s axis, respectively.

The characteristic length L is, in general, defined by the initial conditions. That is, by using a familiar definition of a viscous layer thickness, namely at $z = 0$, $y = \delta_1$, when $u = u_\delta = 0{\cdot}99u_e$, it is readily seen that

$$L = \frac{u_e \delta_{1_c}^2}{\nu}\left[\ln \frac{100(u_e - u_{0_c})}{u_e}\right]^{-1}, \tag{19}$$

where δ_1 is the initial semi-major axis of an isovel of velocity u_δ. With (19), the relation between the physical and transformed axial co-ordinate becomes

$$x = x_c + \frac{u_e \delta_{1_c}^2}{\nu}\left[\ln \frac{100(u_e - u_{0_c})}{u_e}\right]^{-1} s. \tag{20}$$

The physical length x is of the order of a free-stream Reynolds number based on the initial viscous layer thickness of the semi-major axis and, therefore, may be very large.

The distribution of the axial velocity defect for $s > 0$ is derived by the previous analysis and can be shown to be

$$\phi = \frac{u_e - u}{u_e - u_{0_c}} = \phi_0 \exp - k^2[\eta^2 + (\sigma/E)^2], \tag{21}$$

where $k^2 = [1+4s]^{-1}$, $E^2 = [\epsilon^2 + 4s][1+4s]^{-1}$, and ϕ_0 is the distribution of ϕ along the axis and is given by

$$\phi_0 = \frac{\epsilon}{[1+4s]^{\frac{1}{2}}\,[\epsilon^2+4s]^{\frac{1}{2}}}. \tag{22}$$

Equation (21) clearly satisfies the required boundary conditions.

Two important features of the solution are discussed below. The first concerns the various types of modes of decay, while the second deals with the effect of the three-dimensionality on the rate of decay of the velocity defect.

For example, when $\epsilon \ll 1$, that is, when the initial eccentricity of the wake shape is large, equation (22) acquires both two-dimensional and axisymmetric character as s goes from zero to infinity. That is, there exists a region wherein $4s$ is much greater than ϵ^2 and much less than unity; therefore, (22) reduces to $\phi_0 = \epsilon/2s^{\frac{1}{2}}$ which corresponds to a two-dimensional mode of decay. Also, there exists a region wherein $4s \gg 1$, which implies $4s \gg \epsilon^2$, and the mode of decay $(\phi_0 = \epsilon/4s)$ is that of an axially symmetric type. On the other hand, when $4s \gg 1$, which implies $4s \gg \epsilon^2$, the magnitude of ϵ does not enter the discussion and equation (21) reduces to

$$\phi = \frac{\epsilon}{4s}\exp\left\{-\frac{1}{4s}[\eta^2+\sigma^2]\right\}, \tag{23}$$

which has the well-known axisymmetric character.

Equation (23) indicates that a wake with any degree of initial eccentricity ultimately degenerates to an axisymmetric configuration and mode of decay. Clearly, it should be of interest to determine the distance required for the ellipticity to decay to a specified arbitrary value. The viscous layer thicknesses at the principal axes are

$$\text{at}\quad \sigma = 0,\quad \eta_\delta^2 = (1+4s)\ln\left\{\frac{100(u_e-u_0)}{u_e}\right\} = \frac{\delta_1^2 u_e}{L\nu}, \tag{24a}$$

and

$$\text{at}\quad \eta = 0,\quad \sigma_\delta^2 = (\epsilon^2+4s)\ln\left\{\frac{100(u_e-u_0)}{u_e}\right\} = \frac{\delta_2^2 u_e}{L\nu}, \tag{24b}$$

where δ_1 and δ_2 denote the viscous layer thicknesses of the semi-major and semi-minor axes, respectively. With equations (24) it is seen that the ratio of the semi-major to the semi-minor viscous layer thicknesses is

$$\frac{\delta_1}{\delta_2} = \left[\frac{1+4s}{\epsilon^2+4s}\right]^{\frac{1}{2}} \tag{25}$$

and since the eccentricity (e) is defined by

$$e = [1-(\delta_2/\delta_1)^2]^{\frac{1}{2}}, \tag{26}$$

it follows that

$$s = \frac{e_c^2 - e^2}{4e^2}, \tag{27}$$

where $e_c = \sqrt{(1-\epsilon^2)}$.

The question now arises as to the effect of the three-dimensionality on the rate of decay of the velocity defect. A pertinent parameter for comparison is the velocity defect along the axis. If subscripts a and b respectively denote two wakes with different eccentricity, then from (22) it follows that

$$\frac{[(u_e-u_0)/(u_e-u_{0c})]_a}{[(u_e-u_0)/(u_e-u_{0c})]_b} = \frac{\epsilon_a[1+4(x-x_c)/L_b]^{\frac{1}{2}}\,[\epsilon_b^2+4(x-x_c)/L_b]^{\frac{1}{2}}}{\epsilon_b[1+4(x-x_c)/L_a]^{\frac{1}{2}}\,[\epsilon_a^2+4(x-x_c)/L_a]^{\frac{1}{2}}}. \tag{28}$$

It is only meaningful to compare two wakes with equal drag. Therefore, (28) must be supplemented by the condition

$$\theta_{ca} = \theta_{cb}, \tag{29}$$

where θ is the momentum thickness and is defined by

$$\theta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy\,dz. \tag{30}$$

With (5*a*), (18) and (19), condition (29) reduces to

$$\left[\frac{L\nu}{u_e}\epsilon\frac{u_e-u_{0_c}}{u_e}\right]_a = \left[\frac{L\nu}{u_e}\epsilon\frac{u_e-u_{0_c}}{u_e}\right]_b, \tag{31a}$$

or

$$\left[\frac{\delta_{1_c}^2\epsilon(u_e-u_{0_c})/u_e}{\ln\{(u_e-u_{0_c})/u_e\}}\right]_a = \left[\frac{\delta_{1_c}^2\epsilon(u_e-u_{0_c})/u_e}{\ln\{u_e-u_{0_c})/u_e\}}\right]_b. \tag{31b}$$

In order to present some representative results, a special case is considered wherein the flight conditions and the initial velocity at the axis (u_{0_c}) are identical in both wakes. In addition, comparison is made with respect to an axially-symmetric configuration and, therefore, without loss in generality ϵ_b can be set equal to unity. With these assumptions (31*a*) becomes $(L\epsilon)_a = L_b$ and (28) reduces to

$$\frac{(u_e-u_0)_a}{(u_e-u_0)_b} = \frac{[\epsilon_a+\beta]}{[1+\beta]^{\frac{1}{2}}\,[\epsilon_a^2+\beta]^{\frac{1}{2}}}, \tag{32}$$

where $\beta = 4(x-x_c)/L_a$. Investigation of (32) shows that at any station for $x > x_c$ the right-hand side is always less than unity. Therefore, it can be concluded that if two wakes have identical flight conditions, identical initial velocity at the axis, and identical drag [i.e. $\nu_a = \nu_b$, $u_{e_a} = u_{e_b}$, $(u_{0_c})_a = (u_{0_c})_b$ and $(\delta_{1_c}^2\epsilon)_a = (\delta_{1_c}^2\epsilon)_b$], the wake with the largest initial eccentricity ($e_c = (1-\epsilon^2)^{\frac{1}{2}}$) will decay most rapidly.

(*b*) Turbulent flow. For fully turbulent wakes, in which one assumes that the eddy viscosity varies only with the streamwise distance, the previous solutions and results still apply. However, the relation between the physical and transformed axial co-ordinates must be modified to include the eddy coefficient of viscosity. To obtain a qualitative estimate of the change in the streamwise scale due to turbulence, a relation of the following form may be assumed for the eddy coefficient of viscosity:

$$\mu_t = K\rho(2\bar{\delta})\,(u_e-u_0), \tag{33}$$

where K is a pure constant of the order of 10^{-2} and $\bar{\delta}$ is assumed to be the average viscous layer thickness ($2\bar{\delta} = \delta_1+\delta_2$). With (5*a*), (19), (24) and (33) the relation between the physical length x and the transformed axial variable s becomes

$$x = x_c + \frac{u_e\delta_{1_c}}{\epsilon K(u_e-u_{0_c})}[\ln\{100(u_e-u_{0_c})/u_e\}]^{-\frac{1}{2}} \times \int_0^s \frac{[1+4s]^{\frac{1}{2}}\,[\epsilon^2+4s]^{\frac{1}{2}}\,ds}{([1+4s]^{\frac{1}{2}}+[\epsilon^2+4s]^{\frac{1}{2}})\,(\ln\{100(u_e-u_0)/u_e\})^{\frac{1}{2}}}. \tag{34}$$

It is of interest to note that (34) is independent of Reynolds number.

3. Second approximations to the linearized solutions

This section presents two techniques that may be used to improve the analysis; namely, the modified Oseen approximation as suggested by Lewis & Carrier† (1949), and a method wherein the profiles generated by the linear equation are

† The authors wish to thank G. F. Carrier for his useful discussion concerning this approximation.

given an additional degree of freedom in the nature of an undetermined arbitrary function. The arbitrary function is required to satisfy an appropriate condition, such as an integral relation or the axial momentum equation evaluated at the axis.

The modified Oseen approximation involves a corrective stretching of the axial co-ordinate. To this end the inertial term in equation (2) is replaced by $u^*\partial\bar{u}/\partial x$, where $u^* = cu_e$ and c is a constant with values in the range $0 < c < 1{\cdot}0$. Clearly the solution for the elliptic wake is the same as that given in (21) with s replaced by ξ, where $\xi = s/c$. That is,

$$(u_e-u)/(u_e-u_{0_c}) = \tilde{\phi}_0 \exp\{-\tilde{k}^2[\eta^2+(\sigma/\tilde{E})^2]\}, \tag{35}$$

where $\tilde{\phi}_0 = \epsilon[(1+4\xi)(\epsilon^2+4\xi)]^{-\frac{1}{2}}$, $\tilde{k}^2 = [1+4\xi]^{-1}$, and $\tilde{E}^2 = [\epsilon^2+4\xi][1+4\xi]^{-1}$. The constant c may be evaluated in several alternative ways. For example, it may be required that the mean of the neglected inertia terms along the s axis must vanish. This is expressed by

$$\int_{x_e}^{\infty}(u_e-u_ec-\bar{u})\,\bar{u}_x dx = 0, \quad \text{at} \quad y=z=0, \tag{36}$$

from which it follows that $c = (u_e+u_{0_c})/2u_e$. Also, it may be appropriate to weigh the difference by $\bar{u}$. This is expressed by

$$\int_{x_c}^{\infty}\bar{u}(u_e-u_ec-\bar{u})\,\bar{u}_x dx = 0 \quad \text{at} \quad y=z=0, \tag{37}$$

from which we derive $c = (u_e+2u_{0_c})/3u_e$. The two derived magnitudes of c are close to each other.

As an alternative to co-ordinate stretching, one may pose a correction to the linearized solution in the form

$$[(u_e-u)/(u_e-u_{0_c})]_2 = (\phi_0+\phi_1)\exp\{-k^2[\eta^2+(\sigma/E)^2]\}, \tag{38}$$

where $\phi_1(s)$ is an unknown correction function and subscript 2 denotes the second approximation. Several alternative conditions may be used to evaluate this function. Among the possible relations which would be meaningful are the momentum integral equation and the non-linear differential axial momentum equation evaluated at the axis. These equations are respectively†

$$\int_0^{\infty}\int_0^{\infty}\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy\,dz = \left[\int_0^{\infty}\int_0^{\infty}\frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy\,dz\right]_c, \tag{39a}$$

and

$$u_0 u_{0x} = \nu(u_{yy_0}+u_{zz_0}). \tag{39b}$$

It is seen that the momentum defect integral is invariant in the general three-dimensional, steady isobaric case. This can also be shown to be valid for compressible flows.

Only the satisfaction of the integral equation above is shown here because it can be carried out in closed form. The equation along the axis is more cumber-

† The following boundary-layer equations are assumed to apply

continuity: $u_x+v_y+w_z = 0$,

momentum: $(u^2)_x+(uv)_y+(uw)_z = \nu[u_{yy}+u_{zz}]$; $p = p(y, z)$.

some to handle, although it is straightforward in principle. Furthermore, it is possible to satisfy both conditions simultaneously (and indeed others) if additional arbitrary functions are incorporated into the linearized solution. In essence these procedures resemble those of more conventional integral methods, the linearized equation being used to generate first approximation profiles. Equations (38) and (39a) yield

$$\left[\frac{u_e-u}{u_e-u_{0c}}\right]_2 = \left[\frac{u_e}{u_e-u_{0c}}(1-\{1-[1-(u_{0c}^2/u_e^2)]\,\phi_0\}^{\frac{1}{2}})\right]\exp\{-k^2[\eta^2+(\sigma/E)^2]\}, \tag{40}$$

where, without loss in generality, ϕ_{1c} has been taken equal to zero.

Table 1 compares the velocity distribution along the axis given by the linearized solution and by equation (40).

Linearized (ϕ_0)	Second-approximation ($\phi_0+\phi_1$) $u_{0_c} = 0{\cdot}4u_e$	$u_{0_c} = 0{\cdot}6u_e$	$u_{0_c} = 0{\cdot}8u_e$
1·0	1·000	1·000	1·000
0·8	0·712	0·754	0·780
0·5	0·397	0·437	0·472
0·1	0·0717	0·0815	0·0911
0	0	0	0

TABLE 1.

$\phi_0 = \epsilon[(1+4s)(\epsilon^2+4s)]^{-\frac{1}{2}}$ and $\phi_0+\phi_1 = \frac{u_e}{u_e-u_0}(1-\{1-[1-(u_{0_c}^2/u_e^2)]\,\phi_0\}^{\frac{1}{2}})$.

The results of the modified Oseen approximation agree reasonably with the results presented in Table 1.

It is of interest to note that the above corrections do not alter the essential features given by the linearized solution.

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